

The University of Chicago

# PAIRS OF SURFACES IN FIVE-DIMENSIONAL SPACE

A DISSERTATION SUBMITTED TO THE  
FACULTY OF THE DIVISION OF THE  
PHYSICAL SCIENCES IN CANDIDACY FOR  
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1935

By

LEE ROY WILCOX

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## PAIRS OF SURFACES IN FIVE-DIMENSIONAL SPACE

### INTRODUCTION

The purpose of this paper is to construct a theory of the projective differential properties possessed by two analytic surfaces, in the same linear space  $S_5$  of five dimensions, with an analytic one-to-one point correspondence between them. The method used is that of Wilczynski,<sup>1</sup> in which the entire theory is based on a set of differential equations representing the given configuration, power series and differential forms playing a subordinate rôle. The paper is divided into three chapters, dealing respectively with the analytic foundations, the geometrical theory proper, and an associated theory of consecutive configurations.

In the first two sections, a system of six fundamental partial differential equations is derived, integrability conditions are calculated for them, and a canonical form is obtained by a consideration of transformations of analytic representation leaving the geometrical elements unchanged. The differential equations are found in Section 3 to lead directly to local power series expansions representing the two surfaces in neighborhoods of two corresponding points.

At the outset of the geometrical considerations in Section 4 it is found possible to define in a simple manner three one-

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<sup>1</sup>See, for example, Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, Leipzig, 1906.

parameter families of curves on the surfaces. These curves, called hyperasymptotic curves, are found to be of fundamental importance throughout the entire theory. A study in Section 5 of the cones of del Pezzo of the two surfaces suggests the definition of a family of conics called conics of del Pezzo. A number of properties of a family of hyperquadrics having contact of the second order with both surfaces at a pair of corresponding points are developed in Section 6.

The investigations in the third chapter are of a somewhat different character from those preceding. Here certain neighborhood properties of envelopes and loci of some of the spaces geometrically related to the surfaces are studied. This theory centers about a relation, called hyperconjugacy, between pairs of tangents at a point of one of the surfaces, and a net of curves, which is called the characteristic net, on one of the surfaces. The analytic basis for this theory consists of two sets of formulas for the differentiation of local coordinates.



## CHAPTER I

### ANALYTIC FOUNDATIONS

Introduction. In this chapter an analytic basis for a theory of pairs of surfaces in  $S_5$  is constructed. It is found in the first section that an analytic correspondence between the points of two analytic surfaces immersed in a linear space  $S_5$  gives rise, in general to a system of six linear partial differential equations of the second order. Conditions of integrability for these equations are calculated, and in Section 2 a canonical form is obtained. Finally, in the third section, local power series expansions are developed, which represent the two surfaces in neighborhoods of a general pair of corresponding points on them. These expansions are fundamental for the geometrical considerations of the subsequent chapters.

1. The differential equations and integrability conditions. Let us consider two analytic surfaces  $S_x$  and  $S_y$ , immersed in a linear space  $S_5$  of five dimensions, and let us consider further an analytic one-to-one point correspondence between two regions of the respective surfaces. Parametric equations of the surfaces may be written in the form

$$(1) \quad x_i = x_i(u, v), \quad y_i = y_i(u, v) \quad (i = 0, \dots, 5),$$

wherein the  $x_i$  are homogeneous projective coordinates of a variable point on  $S_x$ , and  $x_i(u, v)$  are six complex functions of the two complex variables  $u, v$  which are analytic in a certain

domain;<sup>1</sup> similar statements hold for  $y_1$  and  $y_1(u, v)$ . The same curvilinear coordinates are used for the two surfaces, as may always be done when an analytic correspondence between two surfaces is given.<sup>2</sup> Corresponding points will be assumed throughout to have the same curvilinear coordinates. Equations (1) may be written in vector form

$$(2) \quad x = x(u, v), \quad y = y(u, v).$$

We shall set forth at this point two hypotheses on the surfaces and correspondence. Further restrictions, which cannot be conveniently stated here, are found in Sections 4, 5 and 8.

H 1. Neither surface is an integral surface of a partial differential equation of the form

$$(3) \quad A\phi_{uu} + 2B\phi_{uv} + C\phi_{vv} + 2D\phi_u + 2E\phi_v + F\phi = 0,$$

in which the coefficients are scalar functions of  $u, v$ . Moreover, the fundamental regions of the surfaces are assumed to contain no points at which a relation of the form (3) is conditionally satisfied.<sup>3</sup>

H 2. Tangent planes to the surfaces at each pair of corresponding points do not intersect.

The differential equations which must be satisfied by the vector functions  $x(u, v)$ ,  $y(u, v)$  will now be sought. The points

<sup>1</sup>All functions dealt with in this paper are assumed to be analytic in this domain.

<sup>2</sup>E. P. Lane, Projective Differential Geometry of Curves and Surfaces, Chicago, 1932, pp. 177-8. Corresponding curves, tangent lines and tangent planes have obvious definitions which are described in the reference just given.

<sup>3</sup>See Lane, op. cit., pp. 33-34 and pp. 124-5 for geometrical discussions of this restriction.

$x, x_u, x_v, y, y_u, y_v$ , where subscripts indicate partial differentiation, are everywhere linearly independent, since a linear relation between them implies contradiction of either H 1 or H 2. Hence these points may be regarded as vertices of a coordinate simplex, and all second derivatives of  $x$  and  $y$  are expressible as linear combinations of them in the form

$$\begin{aligned}
 x_{uu} &= ax + bx_u + cx_v + dy + ey_u + fy_v, \\
 x_{uv} &= Ax + Bx_u + Cx_v + Dy + Ey_u + Fy_v, \\
 x_{vv} &= \alpha x + \beta x_u + \gamma x_v + \delta y + \epsilon y_u + \phi y_v, \\
 y_{uu} &= \lambda x + mx_u + nx_v + ry + sy_u + ty_v, \\
 y_{uv} &= Lx + Mx_u + Nx_v + Ry + Sy_u + Ty_v, \\
 y_{vv} &= \lambda x + \mu x_u + \nu x_v + \rho y + \sigma y_u + \tau y_v,
 \end{aligned}
 \tag{4}$$

in which the thirty-six coefficients  $a, \dots, \tau$  are scalar functions of  $u, v$ . In fact, the coefficients may be computed in terms of the functions  $x(u, v)$  and  $y(u, v)$  and their derivatives in the following manner. Substituting in the first of equations (4) the functions

$$x_1(u, v), y_1(u, v) \quad (i = 0, \dots, 5),$$

we obtain six linear equations to be solved for  $a, b, c, d, e, f$ . The determinant  $K$  of this system is given by

$$K = (x, x_u, x_v, y, y_u, y_v),$$

where we have written a typical row in parentheses. Since  $K$  is never zero because of the linear independence of the points  $x, x_u, x_v, y, y_u, y_v$ , there are unique expressions for the quantities  $a, \dots, f$ ; thus, for example,

$$a = \frac{1}{K}(x_{uu}, x_u, x_v, y, y_u, y_v).$$

The remaining coefficients are similarly found.

We turn next to the calculation of integrability conditions for the system (4). Differentiation of the first equation in (4) with respect to  $v$  yields, after  $x_{uv}$ ,  $x_{vv}$ ,  $y_{uv}$ ,  $y_{vv}$  have been replaced by their values given by (4), an expression for  $x_{uuv}$  as a linear combination of  $x$ ,  $x_u$ ,  $x_v$ ,  $y$ ,  $y_u$ ,  $y_v$ . From the second equation in (4)  $x_{uvu}$  is similarly found. Since the vector function  $x(u, v)$  is analytic, we have

$$x_{uuv} = x_{uvu};$$

thus there arises a linear relation among the points  $x$ , ...,  $y_v$ , and by hypothesis H 2 each of the coefficients therein must vanish. Hence we have the six conditions,

$$\begin{aligned} a_v + bA + c\alpha + eL + f\lambda &= A_u + Ba + CA + EX + FL, \\ b_v + c\beta + eM + f\mu &= A + B_u + CB + Em + FM, \\ a + bC + c_v + c\gamma + eN + f\nu &= Bc + C_u + C^2 + En + FN, \\ (5a) \quad bD + c\delta + d_v + eR + f\rho &= Bd + CD + D_u + Er + FR, \\ bE + c\epsilon + e_v + eS + f\sigma &= Be + CE + D + E_u + Es + FS, \\ bF + c\phi + d + eT + f_v + f\tau &= Bf + CF + Et + F_u + FT. \end{aligned}$$

Similarly, from the relations

$$\begin{aligned} x_{uvv} &= x_{vvu}, \\ y_{uuv} &= y_{uvu}, \\ y_{vvu} &= y_{uvv}, \end{aligned}$$

arise eighteen further conditions,



$$\begin{aligned}
 (5b) \quad & A_v + BA + C\alpha + EL + F\lambda = \alpha_u + \beta a + \gamma A + \epsilon \chi + \emptyset L, \\
 & B_v + B^2 + C\beta + EM + F\mu = \alpha + \beta_u + \beta b + \gamma B + \epsilon m + \emptyset M, \\
 & A + BC + C_v + EN + F\nu = \beta c + \gamma_u + \epsilon n + \emptyset N, \\
 & BD + C\delta + D_v + ER + F\rho = \beta d + \gamma D + \delta_u + \epsilon r + \emptyset R, \\
 & BE + C\epsilon + E_v + ES + F\sigma = \beta e + \gamma E + \delta + \epsilon_u + \epsilon s + \emptyset S, \\
 & BF + C\emptyset + D + ET + F_v + F\tau = \beta f + \gamma F + \epsilon t + \emptyset_u + \emptyset T,
 \end{aligned}$$

$$\begin{aligned}
 (5c) \quad & \chi_v + mA + n\alpha + sL + t\lambda = L_u + Ma + NA + S\chi + TL, \\
 & m_v + mB + n\beta + sM + t\mu = L + M_u + Mb + NB + Sm + TM, \\
 & \chi + mC + n_v + n\gamma + sN + t\nu = Mc + N_u + NC + Sn + TN, \\
 & mD + n\delta + r_v + sR + t\rho = Md + ND + R_u + Sn + TR, \\
 & mE + n\epsilon + s_v + t\sigma = Me + NE + R + S_u + TS, \\
 & mF + n\emptyset + r + sT + t_v + t\tau = Mf + NF + St + T_u + T^2,
 \end{aligned}$$

$$\begin{aligned}
 (5d) \quad & L_v + MA + N\alpha + SL + T\lambda = \lambda_u + \mu a + \nu A + \sigma \chi + \tau L, \\
 & M_v + MB + N\beta + SM + T\mu = \lambda + \mu_u + \mu b + \nu B + \sigma m + \tau M, \\
 & L + MC + N_v + N\gamma + SN + T\nu = \mu c + \nu_u + \nu C + \sigma n + \tau N, \\
 & MD + N\delta + R_v + SR + T\rho = \mu d + \nu D + \rho_u + \sigma r + \tau R, \\
 & ME + N\epsilon + S_v + S^2 + T\sigma = \mu e + \nu E + \rho + \sigma_u + \sigma s + \tau S, \\
 & MF + N\emptyset + R + ST + T_v = \mu f + \nu F + \sigma t + \tau_u.
 \end{aligned}$$

There are no further integrability conditions of the system (4) than those listed in (5), since all derivatives of  $x$  and  $y$  have unique expressions when equations (5) are satisfied.

2. Transformations and canonical form of the differential equations. Since the coordinates which we use are homogeneous, introduction of proportionality factors on the functions  $x(u, v)$ ,  $y(u, v)$  leaves the surfaces unaltered. Similarly, the latter are not affected by a change of parameters. The effects on the differential equations (4) of these transformations will be calculated in this section. The computations lead immediately

to a canonical form for the system.

Transformations of proportionality factors are given by

$$(6) \quad \begin{aligned} x &= p(u, v) \bar{x}, \\ y &= q(u, v) \bar{y}, \end{aligned}$$

where  $p$  and  $q$  are scalar functions of  $u, v$  that are nowhere zero. Substituting in (4) the values of  $x$  and  $y$  given by (6), we obtain a new set of differential equations for  $\bar{x}, \bar{y}$  of the same form as (4), in which the new coefficients  $a_1, \dots, \tau_1$  are given by

$$(7) \quad \begin{aligned} pa_1 &= -p_{uu} + ap + bp_u + cp_v, & q\lambda_1 &= \lambda p + mp_u + np_v, \\ pb_1 &= -2p_u + bp, & qm_1 &= mp, \\ pc_1 &= cp, & qn_1 &= np, \\ pd_1 &= dq + eq_u + fq_v, & qr_1 &= -q_{uu} + rq + sq_u + tq_v, \\ pe_1 &= eq, & qs_1 &= -2q_u + sq, \\ pf_1 &= fq, & qt_1 &= tq, \\ pA_1 &= -p_{uv} + Ap + Bp_u + Cp_v, & qL_1 &= Lp + Mp_u + Np_v, \\ pB_1 &= -p_v + Bp, & qM_1 &= Mp, \\ pC_1 &= -p_u + Cp, & qN_1 &= Np, \\ pD_1 &= Dq + Eq_u + Fq_v, & qR_1 &= -q_{uv} + Rq + Sq_u + Tq_v, \\ pE_1 &= Eq, & qS_1 &= -q_v + Sq, \\ pF_1 &= Fq, & qT_1 &= -q_u + Tq, \\ p\alpha_1 &= -p_{vv} + \alpha p + \beta p_u + \gamma p_v, & q\lambda_1 &= \lambda p + \mu p_u + \nu p_v, \\ p\beta_1 &= \beta p, & q\mu_1 &= \mu p, \\ p\gamma_1 &= -2p_v + \gamma p, & q\nu_1 &= \nu p, \\ p\delta_1 &= \delta q + \epsilon q_u + \phi q_v, & q\rho_1 &= -q_{vv} + \rho q + \sigma q_u + \tau q_v, \\ p\epsilon_1 &= \epsilon q, & q\sigma_1 &= \sigma q, \\ p\phi_1 &= \phi q, & q\tau_1 &= -2q_v + \tau q. \end{aligned}$$

Likewise, if a transformation of parameters,

$$(8) \quad \bar{u} = \bar{u}(u, v), \quad \bar{v} = \bar{v}(u, v) \quad (J = \bar{u}_u \bar{v}_v - \bar{u}_v \bar{v}_u \neq 0).$$

be effected on the representation (2), the system (4) becomes a system of the same form, with independent variables  $\bar{u}$ ,  $\bar{v}$  and with new coefficients  $a_2, \dots, \tau_2$ . We record here only the formulas for  $\epsilon_2$  and  $f_2$ , since no use will be made of the expressions for the remaining coefficients:

$$(9) \quad \begin{aligned} J^2 \epsilon_2 &= f \bar{u}_v^3 - (2F - e) \bar{u}_v^2 \bar{u}_u - (2E - \phi) \bar{u}_v \bar{u}_u^2 + e \bar{u}_u^3, \\ J^2 f_2 &= f \bar{v}_v^3 - (2F - e) \bar{v}_v^2 \bar{v}_u - (2E - \phi) \bar{v}_v \bar{v}_u^2 + e \bar{v}_u^3. \end{aligned}$$

A canonical form for equations (4) will now be found.

By direct calculation one obtains from the integrability conditions (5) the relation

$$(b + C + s + T)_v = (B + \gamma + S + \tau)_u;$$

hence there exists a function  $\Pi(u, v)$  such that

$$\Pi_u = b + C + s + T, \quad \Pi_v = B + \gamma + S + \tau.$$

From equations (7) one finds

$$(10) \quad \begin{aligned} b_1 + C_1 + s_1 + T_1 &= -\frac{3pu}{p} - \frac{3qu}{q} + \Pi_u, \\ B_1 + \gamma_1 + S_1 + \tau_1 &= -\frac{3qv}{q} - \frac{3pv}{p} + \Pi_v, \end{aligned}$$

Hence the left members in (10) may be made to vanish if  $p, q$  can be chosen to satisfy

$$(3 \log pq - \Pi)_u = (3 \log pq - \Pi)_v = 0.$$

This is clearly possible since these equations possess the solution

$$pq = ke^{\frac{1}{3}\Pi} \quad (k = \text{const.}).$$

Henceforth it will be supposed that

$$(11) \quad b + C + s + T = B + \gamma + S + \tau = 0.$$

The most general transformation of proportionality factor leaving (11) invariant is (6) with  $pq = \text{const.}$

Further simplification of equations (4) may be effected with the help of a change of parametric nets on the surfaces. From (9) we find that  $\epsilon_2$  and  $f_2$  may be made zero if two independent functions  $\bar{u}(u, v)$ ,  $\bar{v}(u, v)$  can be found satisfying

$$(12) \quad f\theta_v^3 - (2F - e)\theta_v^2\theta_u - (2E - g)\theta_v\theta_u^2 + \epsilon\theta_u^3 = 0.$$

Since the solutions of (12) are identical with those of the factors of the ordinary differential equation<sup>1</sup>

$$(13) \quad fdu^3 + (2F - e) du^2dv - (2E - g) dudv^2 - \epsilon dv^3 = 0,$$

it is easily shown that the existence of the desired functions  $\bar{u}$ ,  $\bar{v}$  is equivalent to the existence of two distinct factors of the left member in (13). Hypothesis H 3 insuring existence of such factors is stated in Section 4, after a geometrical characterization of the curves on  $S_x$  defined by (13) has been given. Throughout the sequel it will be assumed that  $\epsilon = f = 0$ , unless the contrary is stated. This amounts geometrically to supposing that two of the three one-parameter families of curves on each surface defined by (13) have been made parametric. The residual transformation of parameters leaving this parametric net invariant takes the form

$$\bar{u} = U(u), \quad \bar{v} = V(u) \quad (U'V' \neq 0).$$

3. Local power series expansions. Let us consider a point  $X$  on  $S_x$  in a neighborhood of any fixed point  $P_x$  within the fundamental region. Let the curvilinear coordinates of  $P_x$

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<sup>1</sup>A. Cohen, An Elementary Treatise on Differential Equations Boston, 1906, pp. 205-7.



be  $u$ ,  $v$  and those of  $X$  be  $u + \Delta u$ ,  $v + \Delta v$ . Since the surface  $S_x$  is analytic, the coordinates  $X$  are expressible as power series in  $\Delta u$  and  $\Delta v$ ,

$$(14) \quad X = x + (x_u \Delta u + x_v \Delta v) + \frac{1}{2}(x_{uu} \Delta u^2 + 2x_{uv} \Delta u \Delta v + x_{vv} \Delta v^2) + \dots,$$

convergent for  $|\Delta u|$ ,  $|\Delta v|$  sufficiently small. If we select a local coordinate simplex with vertices at the points  $x$ , ...,  $y_v$ , and unit point so that the point

$$(15) \quad x_0 x + x_1 x_u + x_2 x_v + x_3 y + x_4 y_u + x_5 y_v$$

shall have local coordinates proportional to  $x_0$ , ...,  $x_5$ , we find local expansions for  $X$ , which become, after simplification<sup>1</sup> by (4),

$$(16) \quad \begin{aligned} x_0 &= 1 + \frac{1}{2}(a \Delta u^2 + 2A \Delta u \Delta v + \alpha \Delta v^2) + \dots, \\ x_1 &= \Delta u + \frac{1}{2}(b \Delta u^2 + 2B \Delta u \Delta v + \beta \Delta v^2) + \dots, \\ x_2 &= \Delta v + \frac{1}{2}(c \Delta u^2 + 2C \Delta u \Delta v + \gamma \Delta v^2) + \dots, \\ x_3 &= \frac{1}{2}(d \Delta u^2 + 2D \Delta u \Delta v + \delta \Delta v^2) + \dots, \\ x_4 &= \frac{1}{2}(e \Delta u^2 + 2E \Delta u \Delta v + \epsilon \Delta v^2) + \dots, \\ x_5 &= \frac{1}{2}(f \Delta u^2 + 2F \Delta u \Delta v + \phi \Delta v^2) + \dots, \end{aligned}$$

and which converge in the same neighborhood as does the series (14). Similar expansions might be obtained for a point  $Y$  on  $S_y$  near  $P_y$  referred to the same simplex<sup>2</sup>  $\{x, x_u, x_v, y, y_u, y_v\}$ .

From (16) may be found expansions representing an analytic curve on  $S_x$  and passing through  $P_x$ . Let  $C_{x\psi}$  be such a curve, defined analytically as an integral curve of the curvilinear

<sup>1</sup>The expansions in this section are written without the restriction  $\epsilon = f = 0$ .

<sup>2</sup>Braces inclosing a set of points will represent the simplex whose vertices are those points.

differential equation

$$(17) \quad dv - \psi(u, v) du = 0.$$

Then for a point on  $C_x\psi$  with curvilinear coordinates  $u + \Delta u$ ,  $v + \Delta v$ , we have expansions

$$(18) \quad \Delta v = \psi \Delta u + \frac{1}{2} \psi' \Delta u^2 + \dots \quad (\psi' = \psi_u + \psi_v \psi),$$

which converge for  $|\Delta u|$  sufficiently small. To get local power series representing  $C_x\psi$ , we substitute in (16) the value of  $\Delta v$  given by (18). The results, together with the symmetric equations of the curve  $C_y\psi$  on  $S_y$  corresponding to  $C_x\psi$ , are<sup>1</sup>

$$(19) \quad \begin{aligned} x_0 &= 1 + \frac{1}{2}(a + 2A\psi + \alpha\psi^2) \Delta u^2 + \dots, \\ x_1 &= \Delta u + \frac{1}{2}(b + 2B\psi + \beta\psi^2) \Delta u^2 + \dots, \\ x_2 &= \psi \Delta u + \frac{1}{2}(c + 2C\psi + \gamma\psi^2 + \psi') \Delta u^2 + \dots, \\ x_3 &= \frac{1}{2}(d + 2D\psi + \delta\psi^2) \Delta u^2 + \dots, \\ x_4 &= \frac{1}{2}(e + 2E\psi + \epsilon\psi^2) \Delta u^2 + \dots, \\ x_5 &= \frac{1}{2}(f + 2F\psi + \phi\psi^2) \Delta u^2 + \dots, \end{aligned}$$

$$(20) \quad \begin{aligned} y_0 &= \frac{1}{2}(\chi + 2L\psi + \lambda\psi^2) \Delta u^2 + \dots, \\ y_1 &= \frac{1}{2}(m + 2M\psi + \mu\psi^2) \Delta u^2 + \dots, \\ y_2 &= \frac{1}{2}(n + 2N\psi + \nu\psi^2) \Delta u^2 + \dots, \\ y_3 &= 1 + \frac{1}{2}(r + 2R\psi + \rho\psi^2) \Delta u^2 + \dots, \\ y_4 &= \Delta u + \frac{1}{2}(s + 2S\psi + \sigma\psi^2) \Delta u^2 + \dots, \\ y_5 &= \psi \Delta u + \frac{1}{2}(t + 2T\psi + \tau\psi^2 + \psi') \Delta u^2 + \dots. \end{aligned}$$

---

<sup>1</sup>Existence of a region of convergence of the series (19) and (20) is assured by the theory in Goursat-Hedrick, A Course in Mathematical Analysis, Boston, 1904, Vol. I, pp. 397-9.

## CHAPTER II

### FUNDAMENTAL GEOMETRIC CONSIDERATIONS

Introduction. This chapter is concerned with the basic projective geometric properties of the neighborhoods of the second order of a general pair of corresponding points on the respective surfaces. Section 4 contains local equations of the most fundamental of these configurations referred to the simplex defined in Section 3. Hyperasymptotic curves on the surfaces, which play an important rôle throughout the theory, are defined, and several of their properties are deduced. Section 5 deals with the conics of del Pezzo, which are certain plane sections of the cones of del Pezzo of the two surfaces at corresponding points, and also with a two-parameter family of curves on each surface, which are called inflexion curves because of the character of plane curves obtained from them by a certain projection process. Finally, in section 6, a family of hyperquadrics bearing especially interesting relations to the surfaces is introduced. In terms of these hyperquadrics, a set of two-to-one transformations in the pencil of tangent lines at a point of one of the surfaces is defined and studied.

4. Local equations of the fundamental configurations.  
Hyperasymptotic curves. We tabulate first the local equations of some of the more elementary geometrical configurations covariantly related to one or both of the surfaces. Let  $C_{x\psi}$  be a curve<sup>1</sup>

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<sup>1</sup>The following agreement as to symbols will be made. The symbol  $C_{x\psi}$  represents a curve on  $S_x$  with direction  $\psi$ ;  $\Pi_x$

on  $S_x$  passing through  $P_x$ , and let it be regarded as imbedded in the family whose differential equation is

$$dv - \psi(u, v) du = 0.$$

The tangent line  $t_{x\psi}$  to  $C_{x\psi}$  at  $P_x$  is determined by points whose local coordinates are  $1, 0, 0, 0, 0, 0$  and  $0, 1, \psi, 0, 0, 0$ ; hence its local equations are

$$(21) \quad x_2 - \psi x_1 = x_3 = x_4 = x_5 = 0.$$

In similar manner, the osculating plane of  $C_x$  at  $P_x$  is found with the help of expansions (19) to have the equations<sup>1</sup>

$$(22) \quad \frac{x_2 - \psi x_1}{c + (2C - b)\psi - (2B - \gamma)\psi^2 - \beta\psi^3} = \frac{x_3}{d + 2D\psi + \delta\psi^2} \\ = \frac{x_4}{e + 2E\psi} = \frac{x_5}{2F\psi + \theta\psi^2}.$$

The tangent plane  $\Pi_x$  is the locus of lines (21) as  $\psi$  varies.<sup>2</sup> Hence its equations are

$$(23) \quad x_3 = x_4 = x_5 = 0.$$

The space  $S(2, 1)$  of  $S_x$  in the direction  $\psi$  at  $P_x$ , i.e., the locus of the osculating planes of all curves through  $P_x$  having direction

represents the tangent plane of  $S_x$  at  $P_x$ ;  $t_{x\psi}$  represents a tangent line to  $C_{x\psi}$ ;  $\lambda_{xy}$  is the join of two corresponding points  $P_x$  and  $P_y$ . Similar statements hold with  $x$  and  $y$  interchanged.

<sup>1</sup>It is understood that if the denominator of a fraction in the continued equality (22) is zero, that fraction is deleted from (22), and the equation consisting of its numerator set equal to zero is adjoined.

<sup>2</sup>Here  $\psi$  is regarded as varying over the class of functional values of all analytic functions of  $(u, v)$ , corresponding to fixed values of the variables, i.e., the class of all complex numbers.



$\psi$ , is a space  $S_3$  whose equations are found from (22) by elimination of  $\psi'$ . The result is

$$(24) \quad \frac{x_3}{d + 2D\psi + \delta\psi^2} = \frac{x_4}{e + 2E\psi} = \frac{x_5}{2F\psi + \theta\psi^2}.$$

The line  $\lambda_{xy}$ , which joins  $(1, 0, 0, 0, 0, 0)$  to  $(0, 0, 0, 1, 0, 0)$ , has the local equations

$$(25) \quad x_1 = x_2 = x_4 = x_5 = 0;$$

the space  $S_3(\pi_x, P_y)$ , determined by the tangent plane  $\pi_x$  and the point  $P_y$ , is represented by

$$(26) \quad x_4 = x_5 = 0.$$

The point  $P_y$  and the line  $t_{x\psi}$  through  $P_x$  determine a plane whose equations are

$$(27) \quad x_2 - \psi x_1 = x_4 = x_5 = 0;$$

the equation of the hyperplane through  $\pi_y$  and the tangent  $t_{x\psi}$  at  $P_x$  is

$$(28) \quad x_2 - \psi x_1 = 0.$$

The ambient of two corresponding tangent lines  $t_{x\psi}$  and  $t_{y\psi}$  is a space  $S_3$  for all values of  $\psi$ , since otherwise the lines would intersect in a point, which would then lie in both  $\pi_x$  and  $\pi_y$ , contrary to hypothesis H 1. The equations of this space  $S_3(t_{x\psi}, t_{y\psi})$  are

$$(29) \quad x_5 - \psi x_4 = x_2 - \psi x_1 = 0.$$

The locus of the lines  $\lambda_{xy}$  as  $u, v$  vary is a variety  $V_3$ . At a

point  $y + kx$  of a generator this  $V_3$  has a tangent<sup>1</sup>  $S_3$ , whose local equations are

$$(30) \quad x_2 - kx_5 = x_1 - kx_4 = 0.$$

It is to be noted that this space is the  $S_3(\pi_x, P_y)$  when the point of tangency coincides with  $P_x$ . If  $k$  be allowed to vary, the space  $S_3$  represented by (30) generates a cone whose vertex is the line  $\lambda_{xy}$ . Its equation is found by elimination of  $k$  to be

$$(31) \quad x_1x_5 - x_2x_4 = 0.$$

The cone (31) is to be called the canonical cone. It is readily seen that the family of 3-spaces (29) is also a set of generators of the canonical cone. The ambient of a space  $S_3$  of the family (29) and one of the family (30) is a hyperplane,

$$(32) \quad (x_2 - \psi x_1) - k(x_5 - \psi x_4) = 0.$$

Further properties of the canonical cone are developed in Section 6.

It is to be understood that from any theorem deducible from the hypotheses set forth thus far one may obtain a symmetrical theorem by interchanging  $x$  and  $y$  in its statement, since the hypotheses are invariant under this interchange. The substitution accompanying the interchange of  $x$  and  $y$  is the following:

$$(33) \quad \begin{pmatrix} x & a & b & c & d & e & f & A & B & C & D & E & F & \alpha & \beta & \gamma & \delta & \in & \phi & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ y & r & s & t & \lambda & m & n & R & S & T & L & M & N & \rho & \sigma & \tau & \lambda & \mu & \nu & x_3 & x_4 & x_5 & x_0 & x_1 & x_2 \end{pmatrix}.$$

All the equations obtained in this section may be transformed by the substitution (33), the new equations possessing symmetrical

<sup>1</sup>Lane, op. cit., pp. 272-3.

interpretations. We turn now to the proof of a fundamental result.

Theorem 1. There exist on the surface  $S_x$  three one-parameter families of curves of which every curve  $C_{x\psi}$  has at each of its points  $P_x$  the following property: The osculating plane of  $C_{x\psi}$  at  $P_x$  lies in the hyperplane through  $\Pi_x$  (tangent to  $S_x$  at  $P_x$ ) and the line  $t_{y\psi}$  through the point  $P_y$  corresponding to  $P_x$ .

To prove the theorem we write an analytic condition that a curve  $C_{x\psi}$  be such that its osculating plane at  $P_x$  is contained in the hyperplane described. This condition is that the points  $x, x', x''$  be linearly dependent on the points  $x, x_u, x_v, y, y'$ , where the accents denote total differentiation with respect to  $u$  in the direction  $\psi$ . This is equivalent to the vanishing of the determinant

$$(X, x, x_u, x_v, y, y_u + y_v\psi)$$

for

$$\begin{aligned} X &= x, \quad X' = x_u + x_v\psi, \\ X &= x_{uu} + 2x_{uv}\psi + x_{vv}\psi^2 + x_v\psi'. \end{aligned}$$

Since the determinant vanishes identically for the first two values of  $X$ , the only remaining condition is

$$(x_{uu} + 2x_{uv}\psi + x_{vv}\psi^2, x, x_u, x_v, y, y_u + y_v\psi) = 0.$$

If we replace  $x_{uu}, x_{uv}, x_{vv}$  by their values in (4) without assuming  $\epsilon = f = 0$ , we find the differential equation of the desired curves to be (13). The curves of Theorem 1 will be called hyperasymptotic curves.

In order to insure the existence of two hyperasymptotic families forming a proper net, and consequently the existence of distinct factors of (13), we make the following hypothesis:

H 3. At each point of  $S_x$  the three hyperasymptotic tangents are distinct.

We have now a geometrical justification of the canonical form of Section 2. The differential equation of the hyperasymptotics on  $S_x$  becomes, if  $e$  and  $f$  be set equal to zero,

$$(34) \quad du \, dv [(2F - e)du - (2E - \phi)dv] = 0.$$

Hence a corollary to H 3 is  $(2F - e)(2E - \phi) \neq 0$ .

It is to be noted that since hypothesis H 4 is not invariant under interchange of  $x$  and  $y$  we shall no longer be able to treat the two surfaces  $S_x$  and  $S_y$  entirely symmetrically.

5. Conics of del Pezzo. Inflexion curves. As is well known,<sup>1</sup> the locus, as  $\psi$  varies, of the space  $S(2, 1)$  of  $S_x$  in the direction  $\psi$  at  $P_x$  is a cone of the second order whose vertex is the tangent plane  $\pi_x$ . This cone, which is known as the cone of del Pezzo of  $S_x$  at  $P_x$ , cuts the tangent plane  $\pi_y$  corresponding to  $\pi_x$  in a conic  $\Gamma_y$ , which we shall call the conic of del Pezzo in  $\pi_y$ . Interchanging the rôles of the two surfaces, we arrive at the symmetrical conic  $\Gamma_x$  in  $\pi_x$ . It is interesting to note that the conic of del Pezzo may be constructed in a different manner due to the following result:

Theorem 2. Let  $C_x\psi$  be a curve on the surface  $S_x$  through  $P_x$ , and let  $X$  be a point on  $C_x\psi$  near  $P_x$ . The limit of the space  $S_3$  through  $X$  and the tangent plane  $\pi_x$  at  $P_x$  as the point  $X$  approaches  $P_x$  along  $C_x\psi$  is the space  $S(2, 1)$  of  $S_x$  in the direction  $\psi$  at  $P_x$ .

The theorem holds for surfaces in  $S_n$ . The proof consists

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<sup>1</sup>See Lane, op. cit., pp. 256-263, for a discussion of the elementary properties of surfaces in hyperspace.

in finding the limits as  $\Delta u$  approaches zero of the four points

$$x, x_u, x_v, (x_{uu} + 2x_{uv}\psi + x_{vv}\psi^2) \Delta u^2 + \dots,$$

which determine the  $S_3$  through  $\Pi_X$  and a point  $X$  on  $C_{X\psi}$  near  $P_X$ . The first three points remain fixed, while the fourth approaches the point  $x_{uu} + 2x_{uv}\psi + x_{vv}\psi^2$ . Hence the limiting  $S_3$  is the desired  $S(2, 1)$  by the discussion on p. 258 of the work by Lane already cited.

Now from Theorem 2 it follows that if a point  $X$  on  $C_{X\psi}$  be projected from  $\Pi_X$  into  $\Pi_Y$ , and the limit  $Q_{Y\psi}$  of the resulting point as  $X$  approaches  $P_X$  be found, the point  $Q_{Y\psi}$  lies in the space  $S(2, 1)$  of  $S_X$  in the direction  $\psi$  at  $P_X$ , and hence generates the conic of del Pezzo  $\Gamma'_Y$  as  $\psi$  varies. This property enables us to calculate without difficulty the parametric equations of  $\Gamma'_Y$ . Let us consider a point  $X$  near  $P_X$  on  $C_{X\psi}$ ; its local coordinates are given by the expansions (19). The projection of  $X$  from  $\Pi_X$  into  $\Pi_Y$  is found to be the point

$$(0, 0, 0, \{d + 2D\psi + \delta\psi^2\} \frac{\Delta u^2}{2} + \dots, \{e + 2E\psi\} \frac{\Delta u^2}{2} + \dots, \{2F\psi + \phi\psi^2\} \frac{\Delta u^2}{2} + \dots).$$

The limit of this point as  $\Delta u$  approaches zero is the point

$$Q_{Y\psi}(0, 0, 0, d + 2D\psi + \delta\psi^2, e + 2E\psi, 2F\psi + \phi\psi^2),$$

whence the parametric equations of  $\Gamma'_Y$  are

$$\begin{aligned} x_0 &= x_1 = x_2 = 0, \\ x_3 &= d + 2D\psi + \delta\psi^2, \\ x_4 &= e + 2E\psi, \\ x_5 &= 2F\psi + \phi\psi^2, \end{aligned} \tag{35}$$



in which  $\psi$  is the parameter. The transformation carrying lines  $t_x\psi$  into points  $Q_y\psi$  is a projectivity<sup>1</sup>  $\mathcal{P}_{xy}$ , in which corresponding elements are characterized by the same values of  $\psi$ .

We inquire under what conditions  $P_y$  may lie on the conic  $\Gamma'_y$ . In order to answer this question, it is sufficient to eliminate  $\psi$  between the last three equations in (35) and to substitute the coordinates 0, 0, 1, 0, 0, 0 in the resulting equation. In order to show that the elimination can be carried out, we need only prove that the determinant  $\Delta$  given by

$$\Delta = \begin{vmatrix} d & D & \delta \\ e & E & 0 \\ 0 & F & \phi \end{vmatrix}$$

is different from zero.<sup>2</sup> A simple calculation shows

$$(36) \quad (x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x) = -K\Delta,$$

where  $K$  is the determinant of the points  $x, \dots, y_v$ . By hypothesis H 1 the left member in (36) never vanishes; hence  $\Delta$  is everywhere different from zero.

Theorem 3. The point  $P_y$  lies on the conic of del Pezzo  $\Gamma'_y$  tangent to  $S_y$  and  $P_y$  if and only if the invariant  $I$  defined by

$$(37) \quad I = e\phi(e\phi - 4EF)$$

vanishes at that place.<sup>3</sup>

<sup>1</sup>Graustein, Introduction to Higher Geometry, New York, 1930, pp. 274-6.

<sup>2</sup>Ibid. pp. 275, f.

<sup>3</sup>The term place will be used to designate a pair of corresponding points  $P_x, P_y$  and their join  $\lambda_{xy}$ . Thus, places are simply isomorphic with pairs  $u, v$ , and with every place are associated tangent planes  $\pi_x, \pi_y$ , conics  $\Gamma'_x, \Gamma'_y$ , etc.

A brief argument shows that the condition  $I = 0$  implies that  $P_Y$  coincides with  $Q_{Y,0}$ ,  $Q_{Y,\infty}$ , or  $Q_{Y\psi_0}$ , where

$$(38) \quad \psi_0 = \frac{2F - e}{2E - \phi}.$$

Hence  $P_Y$  lies on  $\Gamma_Y$  if and only if  $P_Y$  coincides with one of the three points corresponding in the projectivity  $\rho_{xy}$  to hyperasymptotic tangents at  $P_X$ . Since it is convenient for our purposes to exclude the special case  $I = 0$ , we make the following hypothesis:

H 4. At no place does the point  $P_Y$  lie on the conic of del Pezzo  $\Gamma_Y$  associated with that place.

We consider next the relation between the directions of a tangent  $t_{X\psi}$  and the tangent  $t_{Y\bar{\psi}}$  obtained by joining  $P_Y$  to the point  $Q_{Y\psi}$ . From (35) we find the equations of the desired tangent  $t_{Y\bar{\psi}}$  to be

$$x_5 - \bar{\psi}x_4 = x_0 = x_1 = x_2 = 0,$$

where

$$(39) \quad \bar{\psi} = \frac{2F\psi + \phi\psi^2}{e + 2E\psi}.$$

Equation (39) may be regarded as defining a two-to-one transformation  $\mathcal{I}_X$  of the pencil of tangents at  $P_X$  or  $P_Y$  into itself.

Setting  $\bar{\psi} = \psi$  in (39) we find

Theorem 4. At each place the hyperasymptotic tangents on  $S_X$  are the only invariant tangents in the transformation  $\mathcal{I}_X$  associated with that place.

Equations of the projection of a curve  $C_{X\psi}$  from  $\Pi_Y$  into  $\Pi_X$  will now be found. One obtains equations of the space  $S_3$  through  $\Pi_Y$  and a point  $X$  on  $C_{X\psi}$  near  $P_X$ , then the coordinates of the point of intersection of this  $S_3$  with  $\Pi_X$ , whose locus is the desired curve. The local parametric equations of the projected

curve are

$$\begin{aligned}
 (40) \quad x_0 &= 1 + \frac{1}{2}(a + 2A\psi + \alpha\psi^2) \Delta u^2 + \dots, \\
 x_1 &= \Delta u + \frac{1}{2}(b + 2B\psi + \beta\psi^2) \Delta u^2 + \dots, \\
 x_2 &= \psi \Delta u + \frac{1}{2}(c + 2C\psi + \gamma\psi^2 + \psi') \Delta u^2 + \dots, \\
 x_3 &= x_4 = x_5 = 0,
 \end{aligned}$$

in which  $\Delta u$  is the parameter. It is seen that the curve (40) passes through  $P_x$  and has there as tangent the line  $t_{x\psi}$ . It has at  $P_x$  a point of inflexion if and only if  $C_{x\psi}$  is an integral curve of the differential equation

$$(41) \quad \frac{d^2v}{du^2} = \beta \left( \frac{dv}{du} \right)^3 + (2B - \gamma) \left( \frac{dv}{du} \right)^2 + (b - 2C) \frac{dv}{du} - c.$$

The curves on  $S_x$  defined by (41) will be called inflexion curves. Through every point there is one such curve having there a preassigned tangent. A readily demonstrable characteristic property of inflexion curves will be stated without proof.

Theorem 5. Consider a curve  $C_{x\psi}$  on  $S_x$  and its corresponding curve  $C_{y\psi}$  on  $S_y$ . If at each point  $P_x$  of  $C_{x\psi}$  the hyperplane through the osculating plane of  $C_{x\psi}$  at  $P_x$  and the tangent line to  $C_{y\psi}$  at the point  $P_y$  corresponding to  $P_x$  contains the plane  $\Pi_y$  tangent to  $S_y$  at  $P_y$  then  $C_{x\psi}$  is an inflexion curve, and conversely.

We close the section with a brief discussion of the polar line of  $P_y$  with respect to the conic  $\Gamma'_y$ . Its local equations are found to be

$$\begin{aligned}
 (42) \quad x_0 &= x_1 = x_2 = 0, \\
 ix_3 &+ (-\delta^2ed + 2\delta\delta EF - 2\delta eF^2 + 2\delta eDF)x_4 \\
 &+ (-e^2\delta + 2e\delta FE - 2\delta\delta E^2 + 2e\delta DE)x_5 = 0.
 \end{aligned}$$

The points in which the polar line (42) cuts the conic  $\Gamma'_y$  are the points  $Q_y\psi_1$ ,  $Q_y\psi_2$ , where  $\psi_1$  and  $\psi_2$  are the roots of

$$(43) \quad eF + e\phi\psi + E\phi\psi^2 = 0.$$

The tangents  $t_{x\psi}$  to which these points correspond in the projectivity  $\mathcal{P}_{xy}$  will be called characteristic tangents to  $S_x$  at  $P_x$ . By hypothesis H 4 these tangents are everywhere distinct and hence envelop a proper net, which we shall call the characteristic net on  $S_x$ . A property of the characteristic curves will be developed in Section 8.

6. Hyperquadrics. In this section a covariant family of  $\infty^3$  hyperquadrics is introduced, and in terms of it a generalization of the transformation  $\mathcal{J}_x$ , already discussed in the last section, is defined. Following this, a new characterization of the hyperasymptotic curves is given.

The totality of hyperquadrics containing at a given place the planes  $\Pi_x$  and  $\Pi_y$  comprises an eight parameter family, whose local equation is

$$(44) \quad \lambda_1 x_0 x_3 + \lambda_2 x_0 x_4 + \lambda_3 x_0 x_5 + \lambda_4 x_1 x_3 + \lambda_5 x_1 x_4 \\ + \lambda_6 x_1 x_5 + \lambda_7 x_2 x_3 + \lambda_8 x_2 x_4 + \lambda_9 x_2 x_5 = 0,$$

in which the  $\lambda_i$  are homogeneous parameters.

Theorem 6. A hyperquadric  $Q$  through  $\Pi_x$  and  $\Pi_y$  has contact of the second order with the surface  $S_x$  if and only if  $Q$  contains the space  $S_3(\Pi_y, P_x)$ .

To prove the theorem, we note first that a hyperquadric (44) contains the space  $S_3(\Pi_y, P_x)$  if and only if

$$(45) \quad \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

But a hyperquadric (44) has second order contact with  $S_x$  if and only if (44) is satisfied by the series (19) identically in  $\psi$  to and including terms in  $\Delta u^2$ , i.e., in case

$$\begin{aligned}\lambda_1 d + \lambda_2 e &= 0, \\ \lambda_1 D + \lambda_2 E + \lambda_3 F &= 0, \\ \lambda_1 \delta + \lambda_5 \phi &= 0.\end{aligned}$$

These conditions are equivalent to (45), since the determinant  $\Delta$  is not zero, as was seen in Section 5; hence the theorem follows. The result symmetric to Theorem 6 is true since neither H 3 nor H 4 was used in the proof.

We shall confine our attention here to the  $\infty^3$  hyperquadrics (44) that have second order contact with both  $S_x$  and  $S_y$ . By Theorem 6 and symmetrical considerations, these are the hyperquadrics through  $S_3(P_x, \pi_y)$  and  $S_3(P_y, \pi_x)$ , and hence have the equation

$$(46) \quad a_{14} x_1 x_4 + a_{24} x_2 x_4 + a_{15} x_1 x_5 + a_{25} x_2 x_5 = 0,$$

wherein the  $a_{ij}$  are homogeneous parameters. The configuration represented by (46) for fixed  $a_{ij}$  is in general a cone whose vertex is the line  $\lambda_{xy}$  and whose generators are 3-spaces.

Theorem 7. Each cone (46) cuts  $S_x$  in a curve with a triple point at  $P_x$ . Conversely, corresponding to each triple of directions in  $\pi_x$  at  $P_x$  there is a unique cone of the family (46) cutting  $S_x$  in a curve whose triple point tangents have the given directions.

To prove this, one substitutes in (46) expansions for  $x_1, x_2, x_4, x_5$  given in (19) and notes that the coefficient of  $\Delta u^2$  vanishes identically in  $\psi$ , and that the equation resulting from equating to zero the coefficient of  $\Delta u^3$  is cubic in  $\psi$ .



Conversely, if the triple of directions thus found coincides with a given triple defined by a second cubic equation

$$(47) \quad g_0 + g_1\psi + g_2\psi^2 + g_3\psi^3 = 0,$$

the coefficients in the two equations are proportional. Hence there result four equations expressing the  $g_1$  as linear homogeneous functions of the  $a_{1j}$ . These may be solved, since their determinant is  $I^2$  and is therefore not zero by H 4, yielding

$$(48) \quad \begin{aligned} a_{14} &= \phi I g_0, \\ a_{15} &= 4\phi E^2 g_0 - 2\phi E g_1 + e^2 \phi g_2 - 2e^2 F g_3, \\ a_{24} &= -2\phi^2 E g_0 + e\phi^2 g_1 - 2e\phi F g_2 + 4eF^2 g_3, \\ a_{25} &= e I g_3, \end{aligned}$$

in which the proportionality factor has been suppressed. This completes the proof.

Theorem 8. The unique cone (46) cutting  $S_x$  in a curve whose triple-point tangents are the hyperasymptotic tangents is the canonical cone.

The proof consists in setting

$$(49) \quad g_0 = g_3 = 0, \quad g_1 = 2F - e, \quad g_2 = \phi - 2E$$

in (48), and observing that the corresponding equation (46) reduces to the equation (31) of the canonical cone.<sup>1</sup>

A transformation in the pencil of tangents at  $P_x$  will now be defined. Let us consider an arbitrary triple of directions

<sup>1</sup>It may be further shown that the canonical cone cuts  $S_y$  in a curve whose triple point tangents are the hyperasymptotic tangents to  $S_y$  at  $P_y$ . If the hypothesis H 4' symmetric to H 4 be assumed, however, the following result may be obtained: If a cone of the family (46) cuts either of the surfaces  $S_x$ ,  $S_y$  in a curve whose triple point tangents are hyperasymptotic tangents, it cuts the other in like manner; the unique cone with this property is the canonical cone.

(47) and the unique cone (46) which this triple determines by equations (48). The pole-polar correlation of this cone will be denoted by  $\mathcal{I}_1$ . Its equations are

$$\begin{aligned}\xi_0 = \xi_3 = 0, \quad \xi_1 &= a_{14} x_4 + a_{15} x_5, \quad \xi_2 = a_{24} x_4 + a_{25} x_5, \\ \xi_4 &= a_{14} x_1 + a_{24} x_2, \quad \xi_5 = a_{15} x_1 + a_{25} x_2,\end{aligned}$$

where the proportionality factor has been omitted. The transformation carrying each hyperplane that contains the space  $S_3(\pi_y, P_x)$  into its line of intersection with  $\pi_x$  will be denoted by  $\mathcal{I}_2$ . Now since points in  $\pi_y$  are transformed by  $\mathcal{I}_1$  into hyperplanes through  $S_3(\pi_y, P_x)$ , the product  $\mathcal{I}'_x$  given by

$$\mathcal{I}'_x = \mathcal{P}_{xy} \mathcal{I}_1 \mathcal{I}_2$$

in that order is defined and carries each tangent  $t_{x\psi}$  into the tangent  $t_{x\bar{\psi}}$ , where

$$(50) \quad \bar{\psi} = - \frac{a_0 + a_1 \psi + a_2 \psi^2}{b_0 + b_1 \psi + b_2 \psi^2},$$

and

$$\begin{aligned}a_0 &= e\phi Ig_0, \\ a_1 &= 2e\phi^2 Eg_0 - 4e\phi E F g_1 + 2e^2 \phi F g_2 - 4e^2 F^2 g_3, \\ a_2 &= 4\phi^2 E^2 g_0 - 2e\phi^2 E g_1 + e^2 \phi^2 g_2 - 2e^2 \phi F g_3, \\ b_0 &= -2e\phi^2 E g_0 + e^2 \phi^2 g_1 - 2e^2 \phi F g_2 + 4e^2 F^2 g_3, \\ b_1 &= -4\phi^2 E^2 g_0 + 2e\phi^2 E g_1 - 4e\phi E F g_2 + 2e^2 \phi F g_3, \\ b_2 &= e\phi Ig_3.\end{aligned}$$

The form of (50) tells us that  $\mathcal{I}'_x$  defines in general a two-to-one transformation of the pencil of tangents  $t_{x\psi}$  into itself. It should be noted that  $\mathcal{I}_x$  is a function of the fundamental triple

of directions in terms of which it was defined.<sup>1</sup>

Theorem 9. If the fundamental triple of directions of a transformation  $\mathcal{I}'_x$  is the hyperasymptotic triple,  $\mathcal{I}'_x$  reduces to the transformation  $\mathcal{I}_x$  (Section 5, Eq. (39)), and conversely.

The proof of the first part consists in showing that the equation (50) of the transformation  $\mathcal{I}'_x$  reduces to equation (39) representing  $\mathcal{I}_x$  when  $g_0, \dots, g_3$  have the values given in (49). To prove the converse, it suffices to show that the condition  $\mathcal{I}'_x = \mathcal{I}_x$  implies that the coefficients  $a_0, \dots, b_2$  in (50) are proportional to the corresponding coefficients in (39). From this it follows immediately with the help of hypothesis H 4 that the fundamental triple is the hyperasymptotic triple. A further property of the correspondence  $\mathcal{I}'_x$  is contained in

Theorem 10. Let  $t_x \bar{\psi}$  be any tangent at  $P_x$ , and let the two directions  $\psi_1, \psi_2$  of which  $\bar{\psi}$  is the transform under  $\mathcal{I}'_x$  be found by solving (50) for  $\psi$ . The points  $Q_{y\psi_1}, Q_{y\psi_2}$  on  $\Gamma'_y$  corresponding to  $t_x \psi_1, t_x \psi_2$  under  $\mathcal{P}_{xy}$  are collinear with  $P_y$ . Conversely, to each pair of points  $Q_{y\psi_1}, Q_{y\psi_2}$  on  $\Gamma'_y$  collinear with  $P_y$  corresponds a direction  $\bar{\psi}$  which is the transform of  $\psi_1$  and  $\psi_2$  under  $\mathcal{I}'_x$ .

To prove the first part, it is sufficient to verify by direct calculation that the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ d + 2D\psi_1 + \delta\psi_1^2 & e + 2E\psi_1 & 2F\psi_1 + \phi\psi_1^2 \\ d + 2D\psi_2 + \delta\psi_2^2 & e + 2E\psi_2 & 2F\psi_2 + \phi\psi_2^2 \end{vmatrix}$$

<sup>1</sup>It is to be assumed that the fundamental triple is such that the transformation  $\mathcal{I}'_x$  is non-singular, i.e., that the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

has rank two.

vanishes identically in  $\bar{\psi}$  if  $\psi_1, \psi_2$  are assumed to be roots of (50) regarded as a quadratic equation in  $\psi$ . The converse is seen as follows. Let  $Q_{y\psi_1}$  and  $Q_{y\psi_2}$  be any two points on  $\Gamma'_y$  collinear with  $P_y$ . Let  $\bar{\psi}_1, \bar{\psi}_2$  be the respective transforms of  $\psi_1, \psi_2$  under  $\mathcal{I}'_x$ . Now  $\bar{\psi}_1$  is the transform under  $\mathcal{I}'_x$  of two directions  $\psi_1, \psi'_1$ , and by the first part of the theorem the points  $Q_{y\psi_1}, Q_{y\psi'_1}, P_y$  are collinear. Hence  $Q_{y\psi'_1} = Q_{y\psi_2}$ , and  $\psi'_1 = \psi_2$ . But  $\bar{\psi}_1$  was the transform of  $\psi'_1$  under  $\mathcal{I}'_x$ , whence  $\psi_1 = \psi_2$ .

We close the section with some consequences of Theorem 9.

Let  $\bar{\psi}$  be any direction in  $\Pi_x$  at  $P_x$  and  $\psi_1, \psi_2$  the two directions of which  $\bar{\psi}$  is the transform under  $\mathcal{I}'_x$ . Then the points  $Q_{y\psi_1}, Q_{y\psi_2}$  lie on a tangent  $t_{y\tilde{\psi}}$ , where  $\tilde{\psi}$  is found to have the value

$$(51) \quad \tilde{\psi} = \frac{2F + \delta(\psi_1 + \psi_2)}{2E}$$

From (50), the sum  $(\psi_1 + \psi_2)$  is expressible as a linear fractional function of  $\bar{\psi}$ ; hence equation (51) defines a projectivity  $\mathcal{P}_x$ , which depends, of course, on the fundamental triple of the transformation (50) used in calculating  $(\psi_1 + \psi_2)$ . Imposing the condition that this projectivity be the identity, we find

Theorem 11. The projectivity  $\mathcal{P}'_x$  reduces to the identity if and only if its fundamental triple is the hyperasymptotic triple, i.e., if and only if the transformation  $\mathcal{I}_x$  used in defining  $\mathcal{P}_x$  is the transformation  $\mathcal{I}_x$ .

### CHAPTER III

#### CONSECUTIVE CONFIGURATIONS

Introduction. Loci and envelopes of some of the covariant configurations already defined are to be studied in this chapter. First, in Section 7, two sets of formulas for differentiation of local coordinates are derived; these constitute the analytic basis of the geometrical theory which is constructed in the following section. The theory in Section 8 deals with certain envelopes of the spaces  $S_3(\pi_x, P_y)$  and  $S_4(\pi_x, t_{y\psi})$  and also with varieties  $V_3$  consisting of one parameter families of planes  $S_2(P_y, t_{x\psi})$ . A relation between tangents in the pencil at  $P_x$  is introduced early in the section and is seen to play a fundamental rôle throughout the remainder of the theory. This relation is given the name hyperconjugacy because some of its properties are analogous to those of ordinary conjugacy.

7. Fundamental formulas for differentiation of local coordinates. We shall develop here formulas for two types of differentiation of local coordinates. In the first type, the local coordinates of a given point fixed relative to the two surfaces are regarded as functions of the local coordinate simplex  $\{x, \dots, y_v\}$  to which they are referred. Since the simplex is a function<sup>1</sup> of  $u, v$ , the local coordinates of the fixed point are functions of  $u, v$ . It is the derivatives of

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<sup>1</sup>This is evident since the vertices of the simplex are uniquely determined functions of  $u, v$  when the parametric representation of the surfaces has been definitely chosen.



these functions that are first calculated. The second type of differentiation concerns a two-parameter family of points, one of which is associated with each place of the surfaces. The local coordinates of such a point, referred to the simplex constructed at the place with which the point is associated, are thus functions of  $u, v$ . We may, then, compute the derivatives of these functions.

We proceed now to the derivation of the first set of formulas. Let a point  $P$  which is fixed relative to the surfaces  $S_x, S_y$  have coordinates  $x_0, \dots, x_5$  referred to the simplex  $\{x, \dots, y_v\}$ , and coordinates  $X_0, \dots, X_5$  referred to a neighboring simplex

$$(52) \quad \{X, X_u, X_v, Y, Y_u, Y_v\},$$

where  $X, Y$  are corresponding points on  $S_x, S_y$ , respectively, near  $x, y$ . If the curvilinear coordinates of  $x$  and  $y$  are  $u, v$ , and if the increments of the latter are  $\Delta u, \Delta v$ , we have

$$X = x(u + \Delta u, v + \Delta v), Y = y(u + \Delta u, v + \Delta v),$$

and further,

$$X_u = x_u(u + \Delta u, v + \Delta v),$$

together with similar equations for  $X_v, Y_u, Y_v$ . Since the point  $P$  is fixed, there results

$$(53) \quad x_0x + \dots + x_5y_v = X_0X + \dots + X_5Y_v,$$

wherein the proportionality factor has been suppressed. The points  $X, \dots, Y_v$  are expressible as linear combinations of  $x, \dots, y_v$  by means of their power series expansions in  $\Delta u, \Delta v$ :

$$\begin{aligned}
 X &= x + x_u \Delta u + x_v \Delta v + \dots, \\
 X_u &= x_u + x_{uu} \Delta u + x_{uv} \Delta v + \dots, \\
 X_v &= x_v + x_{uv} \Delta u + x_{vv} \Delta v + \dots, \\
 Y &= y + y_u \Delta u + y_v \Delta v + \dots, \\
 Y_u &= y_u + y_{uu} \Delta u + y_{uv} \Delta v + \dots, \\
 Y_v &= y_v + y_{uv} \Delta u + y_{vv} \Delta v + \dots,
 \end{aligned}
 \tag{54}$$

in which the second derivatives  $x_{uu}$ , ...,  $y_{vv}$  are to be replaced by their values given by the fundamental differential equations (4). By means of these results, the right member of (53) can be made a linear combination of  $x$ , ...,  $y_v$  whose coefficients are linear and homogeneous in the  $X_1$ . These coefficients are thus proportional to  $x_0$ , ...,  $x_5$  by the linear independence of the points  $x$ , ...,  $y_v$ . The equations expressing this relation may be solved, for  $|\Delta u|$ ,  $|\Delta v|$  sufficiently small,<sup>1</sup> yielding

$$\begin{aligned}
 X_0 &= x_0 - (a\Delta u + A\Delta v)x_1 - (A\Delta u + \alpha\Delta v)x_2 - (\lambda\Delta u + L\Delta v)x_4 - (L\Delta u + \lambda\Delta v)x_5 + \dots, \\
 X_1 &= -x_0\Delta u + (1-b\Delta u - B\Delta v)x_1 - (B\Delta u + \beta\Delta v)x_2 - (m\Delta u + M\Delta v)x_4 - (M\Delta u + \mu\Delta v)x_5 + \dots, \\
 X_2 &= -x_0\Delta v - (c\Delta u + C\Delta v)x_1 + (1-C\Delta u - \gamma\Delta v)x_2 - (n\Delta u + N\Delta v)x_4 - (N\Delta u + \nu\Delta v)x_5 + \dots, \\
 X_3 &= -(d\Delta u + D\Delta v)x_1 - (D\Delta u + \delta\Delta v)x_2 + x_3 - (r\Delta u + R\Delta v)x_4 - (R\Delta u + \rho\Delta v)x_5 + \dots, \\
 X_4 &= -(e\Delta u + E\Delta v)x_1 - (E\Delta u)x_2 - x_3\Delta u + (1-s\Delta u - S\Delta v)x_4 - (S\Delta u + \sigma\Delta v)x_5 + \dots, \\
 X_5 &= -(F\Delta v)x_1 - (F\Delta u + \phi\Delta v)x_2 - x_3\Delta v - (t\Delta u + T\Delta v)x_4 + (1-T\Delta u - \tau\Delta v)x_5 + \dots,
 \end{aligned}
 \tag{55}$$

where the expansions are correct to terms of the first degree in the increments. To obtain the desired partial derivatives of  $x_1$  with respect to  $u$ , which will be denoted by  $x_1^{(u)}$ , we set

$\Delta v = 0$  in (55) and compute

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<sup>1</sup>This follows since the determinant is a continuous function of  $\Delta u$ ,  $\Delta v$  which is found to have the value unity when the increments vanish.

$$\lim_{\Delta u=0} \frac{X_1 - x_1}{\Delta u}.$$

The results, together with those obtained by interchanging the rôles of  $u$  and  $v$ , are

$$\begin{aligned}
 x_0^{(u)} &= -ax_1 - Ax_2 - \lambda x_4 - \lambda x_5, \\
 x_1^{(u)} &= -x_0 - bx_1 - Bx_2 - \mu x_4 - Mx_5, \\
 x_2^{(u)} &= -cx_1 - Cx_2 - nx_4 - Nx_5, \\
 x_3^{(u)} &= -dx_1 - Dx_2 - rx_4 - Rx_5, \\
 x_4^{(u)} &= -x_3 - ex_1 - Ex_2 - sx_4 - Sx_5, \\
 x_5^{(u)} &= -Fx_2 - tx_4 - Tx_5, \\
 (56) \quad x_0^{(v)} &= -Ax_1 - \alpha x_2 - \lambda x_4 - \lambda x_5, \\
 x_1^{(v)} &= -Bx_1 - \beta x_2 - \mu x_4 - \mu x_5, \\
 x_2^{(v)} &= -x_0 - Cx_1 - \gamma x_2 - Nx_4 - \nu x_5, \\
 x_3^{(v)} &= -Dx_1 - \delta x_2 - Rx_4 - \rho x_5, \\
 x_4^{(v)} &= -Ex_1 - Sx_4 - \sigma x_5, \\
 x_5^{(v)} &= -x_3 - Fx_1 - \phi x_2 - Tx_4 - \tau x_5,
 \end{aligned}$$

Let us consider now the second type of differentiation.

If  $x_1$  be the six local point coordinates referred to the simplex  $\{x, \dots, y_v\}$  constructed at the place  $(u, v)$ , and if  $x_1(u, v)$  be six analytic functions of  $u, v$ , the equations

$$x_1 = x_1(u, v) \quad (i = 0, \dots, 5)$$

define a two-parameter family of points, one of which is associated with each place  $(u, v)$ . The point associated with a neighboring place  $(u + \Delta u, v + \Delta v)$  has coordinates  $X_1$  given by

$$(57) \quad X_1 = x_1(u + \Delta u, v + \Delta v),$$

referred to the neighboring simplex (52). Since this point is

fixed relative to the surfaces, we may apply equations (55) in solved form to obtain its coordinates  $y_1$  referred to the simplex  $\{x, \dots, y_v\}$ . Thus we obtain equations expressing the  $y_1$  as linear combinations of the  $X_1$ . After replacing each  $X_1$  by its value in (57), we may find the desired derivatives  $x_{1(u)}$  of the coordinates  $x_1$  by setting  $\Delta v = 0$  and computing

$$\lim_{\Delta u=0} \frac{y_1 - x_1(u, v)}{\Delta u}$$

The results, together with those obtained by interchanging the rôles of  $u, v$ , are given by

$$\begin{aligned}
 x_{0(u)} &= x_{0u} & + ax_1 + Ax_2 + \lambda x_4 + Lx_5, \\
 x_{1(u)} &= x_{1u} + x_0 & + bx_1 + Bx_2 + mx_4 + Mx_5, \\
 x_{2(u)} &= x_{2u} & + cx_1 + Cx_2 + nx_4 + Nx_5, \\
 x_{3(u)} &= x_{3u} & + dx_1 + Dx_2 + rx_4 + Rx_5, \\
 x_{4(u)} &= x_{4u} + x_3 & + ex_1 + Ex_2 + sx_4 + Sx_5, \\
 x_{5(u)} &= x_{5u} & + Fx_2 + tx_4 + Tx_5, \\
 (58) \quad x_{0(v)} &= x_{0v} & + Ax_1 + \alpha x_2 + Lx_4 + \lambda x_5, \\
 x_{1(v)} &= x_{1v} & + Bx_1 + \beta x_2 + Mx_4 + \mu x_5, \\
 x_{2(v)} &= x_{2v} + x_0 & + Cx_1 + \gamma x_2 + Nx_4 + \nu x_5, \\
 x_{3(v)} &= x_{3v} & + Dx_1 + \delta x_2 + Rx_4 + \rho x_5, \\
 x_{4(v)} &= x_{4v} & + Ex_1 & + Sx_4 + \sigma x_5, \\
 x_{5(v)} &= x_{5v} + x_3 & + Fx_1 + \phi x_2 + Tx_4 + \tau x_5,
 \end{aligned}$$

wherein the subscripts  $u, v$  without parentheses denote ordinary differentiation.

It is worthy of note that the first type of differentiation is actually a special case of the second type. The distinction between the two types is made since they apply in two different kinds of geometrical problems, the first being used in the

theory of envelopes, and the second in the theory of loci.

8. Envelopes and loci of covariant configurations. We are now ready to construct, with the help of formulas (56) and (58), a theory of envelopes and loci of the various geometrical configurations associated with a place of the surfaces  $S_x, S_y$ . Considerations of the characteristics of the space  $S_3(\pi_x, P_y)$  leads to the definition of hyperconjugacy and an interesting property of the characteristic net on  $S_x$  defined at the close of Section 5. The remaining part of the section shows how the hyperconjugacy relation arises in two other situations. In this section we shall assume an additional hypothesis to be satisfied.

H 5. At every point of the surface  $S_x$  the hyperasymptotic tangents are distinct from the characteristic tangents at that point.

By substituting in the differential equation (43) defining the characteristic directions the values of  $\psi$  corresponding to hyperasymptotic tangents, we find that an analytic condition equivalent to H 5 is the non-vanishing of the invariant  $I_1$  defined by placing

$$I_1 = EF(eE + \phi F - e\phi).$$

The local equations of the space  $S_3(\pi_x, P_y)$ , where  $\pi_x$  is tangent to  $S_x$  at  $P_x$ , and  $P_y$  corresponds to  $P_x$ , were found in Chapter II to be

$$(59) \quad x_4 = x_5 = 0.$$

Let us consider now two corresponding curves  $C_{x\psi}, C_{y\psi}$  through  $P_x, P_y$ , and two corresponding points  $X, Y$  on  $C_{x\psi}, C_{y\psi}$  and near  $P_x, P_y$ , respectively. The space  $S_3(\pi_X, Y)$ , where  $\pi_X$  is tangent



to  $S_X$  at  $X$ , cuts the 3 - space (59) in a line whose limit, as  $X, Y$  approach  $P_X, P_Y$ , respectively, is by definition the characteristic of the space  $S_3(\Pi_X, P_Y)$  for displacement in the direction  $\psi$ . The equations of this characteristic are obtained by adjoining to (59) equations resulting from differentiation of (59) in the direction  $\psi$  by formulas (56). The result, after some simplification, is

$$(60) \quad \begin{aligned} (e + E\psi)x_1 + Ex_2 + x_3 &= 0, \\ F\psi x_1 + (F + \phi\psi)x_2 + \psi x_3 &= 0, \\ x_4 &= x_5 = 0. \end{aligned}$$

The line (60) joins  $P_X$  to the point whose local coordinates are

$$(61) \quad 0, (E - \phi)\psi - F, (F - e)\psi - E\psi^2, eF + e\phi\psi + E\phi\psi^2, 0, 0.$$

The equation of the hyperplane  $S_4(\Pi_Y, t_{X\bar{\psi}})$  is (Ch. II, Eq. 26)

$$x_2 - \bar{\psi}x_1 = 0.$$

Since  $P_X$  lies on this hyperplane, the characteristic (60) will lie on it in case the coordinates (61) satisfy its equation.

This condition may be written in the form

$$(62) \quad F(\psi + \bar{\psi}) + \phi\psi\bar{\psi} - \psi\{e + E(\psi + \bar{\psi})\} = 0.$$

The tangent  $t_{X\bar{\psi}}$  is said to be hyperconjugate to the tangent  $t_{X\psi}$  in case the characteristic of the space  $S_3(\Pi_X, P_Y)$  for displacement in the direction  $\psi$  lies in the hyperplane  $S_4(\Pi_Y, t_{X\bar{\psi}})$ . Thus equation (62) is an analytic criterion for hyperconjugacy. An immediate corollary is that there is a unique tangent  $t_{X\bar{\psi}}$  hyperconjugate to a given tangent  $t_{X\psi}$ , and to each tangent  $t_{X\bar{\psi}}$  there correspond in general two tangents  $t_{X\psi}$  to which  $t_{X\bar{\psi}}$  is hyperconjugate. By setting  $\bar{\bar{\psi}} = \psi$  in (62) we

prove with the help of hypothesis H 5

Theorem 12. The only self-hyperconjugate tangents to  $S_x$  at  $P_x$  are the hyperasymptotic tangents to  $S_x$  at  $P_x$ .

The locus of the characteristic line (60) as  $\psi$  varies is evidently a cone in the space  $S_3(\pi_x, P_y)$  with vertex at  $P_x$ . By elimination of  $\psi$  we find the equations of this cone to be

$$(63) \quad eFx_1^2 + E\phi x_2^2 + x_3^2 + (E+\phi)x_2x_3 + (e+F)x_3x_1 + e\phi x_2x_3 = 0, \\ x_4 = x_5 = 0.$$

The locus of the characteristics of the space  $S(\pi_x, P_y)$  is thus seen to be a cone of the second order with its vertex at  $P_x$ . Evidently the cone (63) cuts  $\pi_x$  in two lines through  $P_x$ . Hence we are led to inquire for what directions of displacement the characteristic (60) lies in  $\pi_x$ .

Theorem 13. The only directions of displacement for which the characteristic line of the space  $S_3(\pi_x, P_y)$  may be tangent to  $S_x$  are the directions of the characteristic tangents at  $P_x$ ; the characteristic line for displacement in the direction of each characteristic tangent at  $P_x$  is the other characteristic tangent.

To prove the first part, we note that a line (60) lies in  $\pi_x$  if and only if the point whose coordinates are given in (60) lies in  $\pi_x$ . But this occurs in case  $\psi$  satisfies the equation (43) defining the characteristic directions. For the proof of the second part, let  $\psi_1$  be a characteristic direction satisfying (43). The characteristic of the space  $S_3(\pi_x, P_y)$  is a tangent  $t_{x\psi_1}$  whose equations are

$$(64) \quad x_2 - \psi_2 x_1 = x_3 = x_4 = x_5 = 0,$$

in which the value of  $\psi_2$  is found by substituting in (64) the coordinates (61) to be given by

$$\psi_2 = \frac{(F - e)\psi_1 - E\psi_1^2}{(E - \phi)\psi_1 - F}.$$

The direction  $\psi_2$  is found to satisfy (43) and is shown with the help of hypothesis H 5 to be distinct from  $\psi_1$ . Hence the theorem follows.

Let us consider next the plane of  $P_Y$  and the tangent  $t_{X\bar{\psi}}$  to a curve  $C_{X\bar{\psi}}$  at  $P_X$ . If  $P_X$  and  $P_Y$  be allowed to describe two corresponding curves  $C_{X\psi}$  and  $C_{Y\psi}$  ( $\psi \neq \bar{\psi}$ ) respectively, the plane  $S_2(P_Y, t_{X\bar{\psi}})$  generates a variety  $V_3$ . We ask under what conditions a generator of this variety associated with a given place may have a focus, or, in other words, may intersect a consecutive generator.<sup>1</sup> The plane  $S_2(P_Y, t_{X\bar{\psi}})$  may be thought of as determined by the points

$$(65) \quad (1, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 1, \bar{\psi}, 0, 0, 0).$$

The plane consecutive to it for displacement in a direction  $\psi$  is determined by the points consecutive to the points (65). Let the local coordinates of the point consecutive to a point  $(x_0, \dots, x_5)$  referred to the same simplex be denoted by  $X_0, \dots, X_5$ . Then we have

$$X_1 = x_1 + \{x_{1(u)} + x_{1(v)}\psi\} du \quad (i = 0, \dots, 5).$$

Applying this formula to the points (65), we find the coordinates of the desired consecutive points. In order that the plane  $S_2(P_Y, t_{X\bar{\psi}})$  intersect its consecutive plane, it is necessary and sufficient that the determinant of the points (65) and their

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<sup>1</sup>See Lane, op. cit., pp. 270-2. The word consecutive will have throughout the remainder of this paper the second meaning described on p. 270 in the reference just cited.

consecutive points vanish. This condition reduces to

$$(66) \quad (\bar{\psi} - \psi) [F(\psi + \bar{\psi}) + \phi \psi \bar{\psi} - \psi \{e + E(\psi + \bar{\psi})\}] = 0.$$

Hence we have established

Theorem 14. Let  $P_x, P_y$  be two corresponding points on  $S_x, S_y$  respectively, and let  $t_{x\bar{\psi}}$  be tangent to a curve  $C_{x\bar{\psi}}$  at  $P_x$ . If  $C_{x\psi}, C_{y\bar{\psi}}$  are two corresponding curves through  $P_x, P_y$  respectively, such that  $C_{x\psi}$  is not tangent to  $C_{x\bar{\psi}}$ , the plane  $S_2(P_y, t_{x\bar{\psi}})$  generates a variety  $V_3$  as  $P_x, P_y$  vary along  $C_{x\psi}, C_{y\bar{\psi}}$ . This plane generator of  $V_3$  has a focus if and only if  $t_{x\bar{\psi}}$  is hyperconjugate to the tangent  $t_{x\psi}$  of  $C_{x\psi}$  at  $P_x$ .

The section will be concluded with a brief study of the characteristics of a hyperplane  $S_4(\pi_x, t_{y\bar{\psi}})$ . The equation of this hyperplane is found by applying the substitution (33) to equation (28) to be

$$(67) \quad x_5 - \bar{\psi} x_4 = 0.$$

To find the equations of the characteristic 3 - space of this hyperplane for displacement in a direction  $\psi$ , one adjoins to (67) the equation obtained by differentiating (67) in the direction  $\psi$  in accordance with the formulas (56). The result is

$$(68) \quad \begin{aligned} & (F\psi - e\bar{\psi} - E\psi\bar{\psi})x_1 + (F + \phi\psi - E\bar{\psi})x_2 + (\psi - \bar{\psi})x_3 \\ & + (t + T\psi - s\bar{\psi} - S\psi\bar{\psi} + \bar{\psi}_u + \bar{\psi}_v\psi)x_4 + (T + \tau\psi - S\bar{\psi} - \sigma\psi\bar{\psi})x_5 = 0, \\ & x_5 - \bar{\psi}x_4 = 0, \end{aligned}$$

This characteristic 3 - space is seen to contain the plane  $\pi_x$  if and only if

$$F\psi - e\bar{\psi} - E\psi\bar{\psi} = F + \phi\psi - E\bar{\psi} = 0.$$

But these conditions may be shown with the help of hypotheses

H 4, H 5 to be equivalent to

$$eF + e\phi\psi + E\phi\psi^2 = 0,$$

$$\bar{\psi} = \frac{2F\psi + \phi\psi^2}{e + 2E\psi};$$

thus we have proved

Theorem 15. The characteristic 3 - space of a hyperplane  
 $S_4(\pi_x, t_{y\bar{\psi}})$  for displacement in a direction  $\psi$  contains the plane  
 $\pi_x$  is and only if  $\psi$  is the direction of a characteristic tangent  
at  $P_x$  and the line  $t_{x\bar{\psi}}$  is the transform of  $t_{x\psi}$  under  $J_x$ .

Let us suppose now that the direction  $\psi$  is not a characteristic direction at  $P_x$ . Then the 3 - space (68) cuts  $\pi_x$  in a line through  $P_x$  whose equations are

$$(69) \quad (F\psi - e\bar{\psi} - E\psi\bar{\psi})x_1 + (F + \phi\psi - E\bar{\psi})x_2 = 0,$$

$$x_3 = x_4 = x_5 = 0.$$

By imposing in turn the condition that the line (69) have the directions  $\psi$  and  $\bar{\psi}$ , we establish the following result:

Theorem 16. The characteristic 3 - space of a hyperplane  
 $S_4(\pi_x, t_{y\bar{\psi}})$ , for displacement in a direction  $\psi$  which is not a  
characteristic direction at  $P_x$ , cuts the plane  $\pi_x$  in a line  
through  $P_x$ . This line is the tangent  $t_{x\psi}$  in case  $t_{x\bar{\psi}}$  is the  
transform of  $t_{x\psi}$  under  $J_x$ , and is the tangent  $t_{x\bar{\psi}}$  in case  $t_{x\psi}$   
is hyperconjugate to  $t_{x\bar{\psi}}$ .



## VITA

Lee Roy Wilcox was born in Chicago, Illinois, on June 8, 1912. He attended the public schools in Chicago, Wilmette, and Winnetka, Illinois, and was graduated from the New Trier Township High School in June, 1929. His undergraduate work was done at the University of Chicago during the years 1929-32; he received the degree of Bachelor of Science with honors in mathematics in December, 1932. During the winter, spring and autumn quarters of the years 1933 and 1934, and the winter and spring quarters of the year 1935, he did graduate work at the University of Chicago under Professors Lane, Bliss, Dickson, Lunn, MacMillan, Graves, Logsdon, Bartky and Albert; he received the degree of Master of Science in December, 1933. For the two years 1933-4, 1934-5 he held fellowships in the Department of Mathematics, serving as instructor in elementary courses during the spring quarters, 1934 and 1935. He wishes to acknowledge his indebtedness to Professor Lane for valuable suggestions and criticisms throughout the preparation of this dissertation.



